

FLOWS ON SOLENOIDS ARE GENERICALLY NOT ALMOST PERIODIC

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ABSTRACT. The space of non-singular flows on the solenoid Σ_N is shown to contain a dense G_δ consisting of flows which are not almost periodic. Whether this result carries over to Hamiltonian flows remains an open question.

INTRODUCTION

For any compact symplectic manifold M of dimension at least 4, L. Markus and K. R. Meyer demonstrate in [MM] that the space $\mathfrak{H}^k(M)$ of all C^k ($k \geq 4$) Hamiltonians on M contains a generic subset \mathfrak{M}_Σ such that for each Hamiltonian $dH^\#$ in \mathfrak{M}_Σ and for each solenoid Σ_N there exists a minimal set for the flow induced by $dH^\#$ that is homeomorphic to Σ_N . They leave open the question of whether the flows on these solenoids are almost periodic ([MM], p. 90).

Corresponding to a sequence of natural numbers $N = (n_1, n_2, \dots)$ with $n_j \geq 2$ for each j there is the solenoid Σ_N which is the inverse limit of the inverse sequence $\{X_j, f_j\}_{j=1}^\infty$ with factor space $X_j = S^1 = \{\exp(2\pi it) \in \mathbb{C} \mid t \in [0, 1)\}$ for each j and bonding maps $f_j(z) = z^{n_j}$

$$S^1 \xleftarrow{n_1} S^1 \xleftarrow{n_2} S^1 \xleftarrow{n_3} \dots \quad \Sigma_N = \left\{ \langle z_j \rangle_{j=1}^\infty \in \prod_{i=1}^\infty S^1 \mid z_j = f_j(z_{j+1}) \text{ for } j = 1, 2, \dots \right\}.$$

Here $\prod_{i=1}^\infty S^1$ is the compact topological group with the group operation (written “+”) given by factor-wise multiplication and with the metric

$$d\left(\langle x_j \rangle_{j=1}^\infty, \langle y_j \rangle_{j=1}^\infty\right) = \sum_{j=1}^\infty \frac{1}{2^j} |x_j - y_j|.$$

This metric also serves as a metric for the subgroup Σ_N . There is then the continuous (but not bicontinuous) isomorphism $\pi_N : \mathbb{R} \rightarrow C_N$ onto the Σ_N -arc component C_N of the identity $e = (1, 1, \dots)$ given by

$$t \mapsto \left\langle \exp(2\pi it), \exp\left(\frac{2\pi it}{n_1}\right), \dots, \exp\left(\frac{2\pi it}{n_1 \cdots n_j}\right), \dots \right\rangle,$$

and we have the family of linear flows Λ_N on Σ_N :

$$\begin{aligned} \Lambda_N &= \{\phi_N^\alpha : \mathbb{R} \times \Sigma_N \rightarrow \Sigma_N \mid \alpha \in \mathbb{R}\} \\ \phi_N^\alpha(t, x) &= \pi_N(\alpha t) + x \end{aligned}$$

and any almost periodic flow on Σ_N is equivalent (C^0 conjugate) to some $\phi_N^\alpha \in \Lambda_N$ [C]. (A *flow* is a continuous group action of $(\mathbb{R}, +)$.)

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Given any point $x \in \Sigma_N$ the map $\pi_N + x$ sending $t \mapsto \pi_N(t) + x$ parameterizes the arc component of x , which will coincide with the trajectory of x for any non-singular flow ϕ on Σ_N . For $k = 0, 1, \dots, \infty$ and $0 \leq \alpha \leq 1$, we consider a flow $\phi : \mathbb{R} \times \Sigma_N \rightarrow \Sigma_N$ to be $C^{k+1+\alpha}$ if the associated “vector field”

$$v_\phi : \Sigma_N \rightarrow \mathbb{R}; \quad x \mapsto \frac{d \left[(\pi_N + x)^{-1} \phi(t, x) \right]}{dt} \Big|_{t=0}$$

exists and satisfies the conditions:

1. for $j = 1, \dots, k$ the function $v_\phi^j : \Sigma_N \rightarrow \mathbb{R}; \quad x \mapsto \frac{d^j \left[(\pi_N + x)^{-1} \phi(t, x) \right]}{dt^j} \Big|_{t=0}$ is continuous [$v_\phi^1 = v_\phi$]
2. for each $x \in \Sigma_N$ the function $\mathbb{R} \rightarrow \mathbb{R}; \quad t \mapsto v_\phi^k(x + \pi_N(t))$ is α -Hölder (see below) if $\alpha > 0$ and $k < \infty$.

We then endow the space $C^{k+\alpha}(N)$ of vector fields v_ϕ on Σ_N stemming from $C^{k+1+\alpha}$ flows ϕ with the metric

$$d_k(v_\phi, v_\psi) \stackrel{\text{def}}{=} \sum_{j=1}^k \max_{x \in \Sigma_N} \left| v_\phi^j(x) - v_\psi^j(x) \right| \quad \text{for } 1 \leq k < \infty$$

$$\text{and } d_\infty(v_\phi, v_\psi) \stackrel{\text{def}}{=} \sum_{r=1}^{\infty} \frac{2^{-r} d_r(v_\phi, v_\psi)}{1 + d_r(v_\phi, v_\psi)}.$$

If Σ_N is embedded in some compact C^k manifold so that the flow ϕ_N^1 extends to a C^k flow on the manifold, then this metric induces the same topology as the restriction of the Whitney topology (see [MM], pp. 34–35 for a description of the Whitney topology). We are interested in the non-singular flows on Σ_N and so will work with $C^{k+\alpha}(N+)$, the subspace of $C^{k+\alpha}(N)$ consisting of positive v_ϕ . (Whenever ϕ is non-singular the map v_ϕ must be either positive or negative since each trajectory is dense and the intermediate value theorem yields a singularity if there is a change of signs, and so we consider positive v_ϕ without loss of generality.)

If $p_j : \Sigma_N \rightarrow S^1$ denotes the projection onto the j^{th} factor $\langle x_k \rangle_{k=1}^\infty \mapsto x_j$ and if \widehat{N} denotes the subgroup of $(\mathbb{R}, +)$ generated by

$$\left\{ \frac{1}{n_1}, \frac{1}{n_1 n_2}, \dots, \frac{1}{n_1 n_2 \cdots n_j} \mid j = 1, 2, \dots \right\}$$

and if for $r = \frac{m}{n_1 n_2 \cdots n_j} \in \widehat{N}$ $\chi_r : \Sigma_N \rightarrow S^1$ denotes the homomorphism sending

$x \mapsto (p_{j+1}(x))^m$, then $\Xi(N) = \left\{ \chi_r \mid r \in \widehat{N} \right\}$ is the group of characters of Σ_N .

Given a continuous function $f : \Sigma_N \rightarrow \mathbb{C}$, the theorem of Peter and Weyl guarantees that f can be uniformly approximated by finite linear combinations of characters χ_r (see, e.g., [P]), and since $\chi_{\frac{m}{n}} \circ \pi_N(t) = \exp(2\pi i m t / n)$ the map $f_e : \mathbb{R} \rightarrow \mathbb{C}; \quad t \mapsto f(\pi_N(t))$ is the uniform limit of periodic maps and is thus *limit periodic* by definition (see, e.g., [Be] 1§6), which is a special type of almost periodic function. It then follows that the Bohr–Fourier series

$$f_e(t) \sim \sum_{r \in \widehat{N}} f_r \chi_r \circ \pi_N(t) = \sum_{r \in \widehat{N}} f_r \exp(2\pi i r t)$$

when appropriately ordered converges uniformly to f_e provided f_e is α -Hölder for some $0 < \alpha \leq 1$:

$$\sup_t |f_e(t + \delta) - f_e(t)| < C\delta^\alpha \text{ (for some } C > 0 \text{ and all } \delta > 0) \text{ (see [Be] 1§8)}.$$

Whenever we are dealing with such a function we will assume without comment that the series is so arranged that $f = \sum_{r \in \hat{N}} f_r \chi_r$.

We shall show that in each space $C^{k+\alpha}(N+)$ the collection of vector fields v_ϕ corresponding to flows ϕ which are not almost periodic contains a dense G_δ provided $k \geq 1$ or $\alpha = 1$ (in other words: v_ϕ is Lipschitz).

1. PROOF OF THE MAIN RESULT

Fix a space $C^{k+\alpha}(N+)$ with $k \geq 1$ or $\alpha = 1$. As already mentioned, a flow ϕ on Σ_N is almost periodic exactly when there is a homeomorphism $h' : \Sigma_N \rightarrow \Sigma_N$ providing an equivalence of ϕ with some linear flow $\phi_N^{\alpha'}$, $h' \circ \phi(t, x) \equiv \phi_N^{\alpha'}(t, h'(x))$. Any such h' is homotopic to an automorphism a of Σ_N , and the automorphism a^{-1} will in turn equate $\phi_N^{\alpha'}$ with a linear flow ϕ_N^α (see [S] and [C]). And so ϕ is almost periodic exactly when there is a homeomorphism $h = a^{-1} \circ h'$ homotopic to the identity providing an equivalence between ϕ and a linear flow ϕ_N^α .

We now translate the existence of such an h into a more useful form, so assume ϕ is almost periodic and that such an h exists. Each $x \in \Sigma_N$ belongs to the section $\mathcal{S}_x \stackrel{\text{def}}{=} p_1^{-1}(p_1(x))$ (which is a Cantor set), and the time it takes the linear flow ϕ_N^α to return to \mathcal{S}_x is the constant $\frac{1}{\alpha}$ since

$$p_1(\phi_N^\alpha(t, x)) = p_1(\pi_N(t\alpha) + x) = p_1(\pi_N(t\alpha)) \cdot p_1(x) = \exp(2\pi i t\alpha) \cdot p_1(x)$$

and the smallest positive value of t satisfying $\exp(2\pi i t\alpha) = 1$ is $\frac{1}{\alpha}$. Since $1/x$ is C^∞ on any closed interval not containing 0, $\lambda(x) \stackrel{\text{def}}{=} \frac{1}{v_\phi(x)} = \sum_{r \in \hat{N}} \lambda_r \chi_r(x)$ is $C^{k+\alpha}$ and the ϕ -return time τ of x to the section \mathcal{S}_x is given by

$$\begin{aligned} \tau(x) &= \int_0^1 \lambda(x + \pi_N(t)) dt \\ &= \lambda_0 + \sum_{r \in \hat{N} - \{0\}} \frac{\lambda_r}{2\pi i r} \chi_r(x + \pi_N(1)) - \sum_{r \in \hat{N} - \{0\}} \frac{\lambda_r}{2\pi i r} \chi_r(x) \\ &\sim \lambda_0 + \sum_{r \in \hat{N} - \{0\}} \frac{\lambda_r}{2\pi i r} [\chi_r(\pi_N(1)) - 1] \chi_r(x) = \sum_{r \in \hat{N}} \tau_r \chi_r(x). \end{aligned}$$

Now h is homotopic to the identity and so we have the map $\delta \sim \sum_{r \in \hat{N}} \delta_r \chi_r : \Sigma_N \rightarrow \mathbb{R}$

$$\delta(x) \stackrel{\text{def}}{=} \text{the unique time } t \text{ satisfying } h(x) = \phi_N^\alpha(t, x)$$

since $h(x)$ must lie in the same path component as x . Since h provides a flow equivalence, the ϕ_N^α -return time to the ϕ_N^α -section $h(\mathcal{S}_x)$ for $h(x)$ is $\tau(x)$. We can now express the constancy of the ϕ_N^α -return time to \mathcal{S}_x as

$$\delta(x) + \tau(x) - \delta(x + \pi_N(1)) = \frac{1}{\alpha}$$

or

$$\tau(x) - \frac{1}{\alpha} = \delta(x + \pi_N(1)) - \delta(x).$$

We also have

$$\delta(x + \pi_N(1)) - \delta(x) \sim \sum_{r \in \hat{N} - \{0\}} [\chi_r(\pi_N(1)) - 1] \delta_r \chi_r(x)$$

leading to the conditions

$$\delta_r = \frac{\tau_r}{[\chi_r(\pi_N(1)) - 1]} = \frac{\lambda_r}{2\pi i r} \text{ for } r \in \hat{N} - \{0\}$$

and

$$\lambda_0 = \tau_0 = \frac{1}{\alpha}.$$

(There is no restriction on δ_0 since we may follow h by translations of elements in C_N and obtain other flow equivalences homotopic to the identity.)

We then have the limit periodic function

$$\delta_e(t) \sim \sum_{r \in \hat{N}} \delta_r \exp(2\pi i r t) = \delta_0 + \sum_{r \in \hat{N} - \{0\}} \frac{\lambda_r}{2\pi i r} \exp(2\pi i r t).$$

According to Bohr's theorem on the integral of an almost periodic function (see [B] §68–69), the integral $F(T) = \int_0^T f(t) dt$ of a limit periodic function $f(t) \sim \sum_{r \in \hat{N}} f_r \exp(2\pi i r t)$ is almost periodic if and only if F is bounded, in which case

$$F(t) \sim \sum_{r \in \hat{N}} F_r \exp(2\pi i r t) = F_0 + \sum_{r \in \hat{N} - \{0\}} \frac{f_r}{2\pi i r} \exp(2\pi i r t).$$

Comparing the Fourier–Bohr series of δ_e with that of λ_e , we see that the bounded function $\delta_e(t)$ represents an integral of $\lambda_e(t) - \lambda_0$. Moreover, whenever $\lambda_e(t) - \lambda_0$ has a bounded integral we are able to construct a map δ as above, which in turn allows us to construct an equivalence h . Notice that by its construction δ will be as smooth as λ . This gives us the following result.

Theorem 1.1. *The flow $\phi \in C^{k+\alpha}(N+)$ with vector field v_ϕ is almost periodic if and only if $\lambda_e(t) - \lambda_0$ has a bounded integral, where $\lambda_e(t) = \frac{1}{v_\phi \circ \pi_N(t)} = \sum_{r \in \hat{N}} \lambda_r \exp(2\pi i r t)$.*

If $f(t)$ is periodic with Fourier series $\sum_{r \in \hat{N}} f_r \exp(2\pi i r t)$, a necessary and sufficient condition that it have a periodic integral is that

$$f_0 = M\{f\} \stackrel{\text{def}}{=} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t) dt = 0 \text{ (see [B] §68),}$$

and since any function $\lambda_e(t) - \lambda_0$ as above can be approximated (relative to d_k) arbitrarily closely by a periodic “partial series” (possibly containing infinitely many terms) of its Fourier–Bohr series and since $M\{f\} = 0$ for any such partial series f , we obtain the following.

Corollary 1.2. The collection of almost periodic flows in $C^{k+\alpha}(N+)$ is dense.

(We are making use of the fact that the function $v_\phi \mapsto 1/v_\phi = \lambda$ is a homeomorphism of $C^{k+\alpha}(N+)$.) We postpone the existence of flows for given functions until the next section.

Now we examine the functions $\lambda = 1/v_\phi$ corresponding to flows ϕ which are not almost periodic. First, we note

$$|M\{f\} - M\{g\}| \leq \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |f(t) - g(t)| dt \leq \sup |f(t) - g(t)|.$$

And so $d_1(\lambda - \lambda_0, \mu - \mu_0) \leq 2d_1(\lambda, \mu)$. Now for $n = 1, 2, \dots$ we define the sets

$$U_n \stackrel{\text{def}}{=} \left\{ \lambda \mid \text{there is a } T_n \text{ with } \left| \int_0^{T_n} (\lambda_e(t) - \lambda_0) dt \right| > n \right\}$$

and claim that U_n is open for $n = 1, 2, \dots$. So suppose then that $\lambda \in U_n$ with

$$\delta = \left| \int_0^{T_n} (\lambda_e(t) - \lambda_0) dt \right| - n > 0.$$

Now if $d_1(\lambda, \mu) \leq d_k(\lambda, \mu) < \frac{\delta}{3|T_n|}$ we have

$$\left| \int_0^{T_n} (\lambda_e(t) - \lambda_0) dt - \int_0^{T_n} (\mu_e(t) - \mu_0) dt \right| \leq \left| \int_0^{T_n} \frac{2\delta}{3|T_n|} dt \right| < \delta$$

and so μ is also in U_n , demonstrating that U_n is open. And so the $G_\delta \cap_{n=1}^\infty U_n$ is the collection of $\lambda = 1/v_\phi$ corresponding to flows ϕ which are not almost periodic.

It then remains to show that $\cap_{n=1}^\infty U_n$ is dense. Let $\lambda = 1/v_\phi$ be given. We need to find an arbitrarily close function which corresponds to a flow which is not almost periodic. Consider for $m = 1, 2, \dots$ the C^∞ maps $\Sigma_N \rightarrow \mathbb{R}$

$$\rho_m(x) \stackrel{\text{def}}{=} \sum_{j=m}^\infty \frac{1}{n_1 \cdots n_j} \chi_{\frac{1}{n_1 \cdots n_j}}(x).$$

Bohr's theorem shows that $\rho_m \circ \pi_N(t)$ has an unbounded integral since Parseval's equation for almost periodic functions would fail for the Fourier-Bohr series of a bounded $\int_0^T \rho_m \circ \pi_N(t) dt$, implying that at least one of $\text{Re}(\rho_m \circ \pi_N(t))$ and $\text{Im}(\rho_m \circ \pi_N(t))$ too has an unbounded integral. (Notice that both the real and imaginary parts of a function with mean value 0 also have mean value 0.) If $\lambda_e(t) - \lambda_0$ has unbounded integral, there is nothing to prove; and if not, we can choose $\lambda + \text{Re}\rho_m$ and $\lambda + \text{Im}\rho_m$ positive and as close to λ as desired by choosing m large enough since $|\rho_m(x)| \leq \frac{1}{2^{m-1}}$ (and similarly for the derivatives), and at least one of these two functions (say λ^m) will be such that $\lambda^m \circ \pi_N$ will have an unbounded integral when its mean value λ_0 is subtracted.

Corollary 1.3. The collection of flows in $C^{k+\alpha}(N+)$ which are not almost periodic is a dense G_δ .

2. REALIZATION OF A FLOW FOR A GIVEN VECTOR FIELD

Let v be a positive Lipschitz function $\Sigma_N \rightarrow \mathbb{R}$. We seek a flow ϕ_v on Σ_N which has v as its vector field. For $n \geq 2$ there is a C^∞ flow ϕ on the tube $\Pi_0 = B^{2n-1} \times S^1$ (B^{2n-1} is the closed unit ball of \mathbb{R}^{2n-1}) satisfying:

1. ϕ has a homeomorphic copy \mathfrak{S}_N of Σ_N as a limit set
2. $d\psi/dt \equiv 1$ along all the orbits of ϕ , where $0 \leq \psi < 1$ parameterizes the S^1 factor of Π_0 (see [MM], p. 87).

The limit set \mathfrak{S}_N is realized as the nested intersection of compact tubes Π_j , $j = 0, 1, 2, \dots$, satisfying for $j = 0, 1, \dots$:

1. Π_j is homeomorphic to Π_0 and
2. Π_{j+1} encircles Π_j n_{j+1} times.

Therefore, ϕ restricted to \mathfrak{S}_N has the constant return time of 1 to the sections

$$\mathcal{S}_t \stackrel{\text{def}}{=} \{(x, \psi) \in B^{2n-1} \times S^1 \cap \mathfrak{S}_N \mid \psi = t\}$$

and is thus equivalent to the linear flow ϕ_N^1 on Σ_N via a homeomorphism, say h , where without loss of generality $\psi(h^{-1}(e)) = 0$. Associated with the flow ϕ is a vector field \mathfrak{v} . We shall obtain the desired flow ϕ_v by a time change of the flow ϕ : we shall multiply the vectors of \mathfrak{v} by a function to change their lengths but not their directions and then use h to obtain ϕ_v . (Here we are measuring lengths of vectors by the lengths of their projections onto the \mathbb{R} factor in the tangent bundle corresponding to the S^1 factor in Π_0 .)

Setting $\psi_0 = \psi$, for $j = 1, 2, \dots$ let $0 \leq \psi_j < n_1 \cdots n_j$ parameterize the S^1 factor of Π_j in such a way that for $(x, \psi_j) = (x, \psi_{j-1})$ in Π_j we have $\psi_j = \psi_{j-1} \pmod{1}$ and $\psi_j(h^{-1}(e)) = 0$. (We could use ϕ -sections transverse to the elliptic periodic orbits $\gamma_0, \gamma_1, \dots$ for example (see [MM], p. 87).) For $j = 0, 1, 2, \dots$ let $s_j(t)$ be a purely periodic Lipschitz partial sum of $v_e(t) = \lim_{j \rightarrow \infty} s_j(t) = \sum_{r \in \mathbb{N}} v_r \exp(2\pi i r t)$,

where the period of s_0 is 1 and the period of s_j is $n_1 \cdots n_j$ for $j = 1, 2, \dots$ (see [Be] 1§8). We then form for $j = 0, 1, 2, \dots$ the following functions $\tau_j : \Pi_0 \rightarrow \mathbb{R}$; $\tau_0((x, \psi_0)) = s_0(\psi_0)$, $\tau_1((x, \psi_0)) = \tau_0((x, \psi_0))$ for (x, ψ_0) in $\Pi_0 - \mathcal{T}_1$ (where \mathcal{T}_1 is a tube satisfying $\Pi_1 \subset \mathcal{T}_1 \subset \Pi_0$) and $\tau_1((x, \psi_0))$ is $\tau_0((x, \psi_0))$ gradually changed in $\mathcal{T}_1 - \Pi_1$ until finally $\tau_1((x, \psi_1)) = s_1(\psi_1)$ for $(x, \psi_1) \in \Pi_1$. Continue in the same manner so that $\tau_{j+1}((x, \psi_0)) = \tau_j((x, \psi_0))$ for (x, ψ_0) in $\Pi_0 - \mathcal{T}_{j+1}$ (where \mathcal{T}_{j+1} is a tube satisfying $\Pi_{j+1} \subset \mathcal{T}_{j+1} \subset \Pi_j$) and $\tau_{j+1}((x, \psi_0))$ is $\tau_j((x, \psi_0))$ gradually changed in $\mathcal{T}_{j+1} - \Pi_{j+1}$ until finally $\tau_{j+1}((x, \psi_{j+1})) = s_{j+1}(\psi_{j+1})$ for $(x, \psi_{j+1}) \in \Pi_{j+1}$. And then with $\tau((x, \psi_0)) \stackrel{\text{def}}{=} \lim_{j \rightarrow \infty} \tau_j((x, \psi_0))$ we have the function to obtain the desired time change of \mathfrak{v} .

This also shows that for arbitrarily small time changes we can alter a generic class of Hamiltonian flows to obtain flows which are not Lyapunov stable on solenoidal minimal sets (see [NS] V§8). By adjusting the coefficients of the ρ_m as in the previous section, we even have flows which have points on the same orbit arbitrarily close that “lap” one another relative to these tubes. However, it is not clear what happens if we restrict ourselves to Hamiltonian flows. To make matters even more complicated, it is not clear that these solenoids persist as limit sets under small perturbations (see [AM] 8§5).

REFERENCES

- [AM] R. Abraham and J. E. Marsden, *Foundations of mechanics*, The Benjamin/Cummings Publishing Comp., Inc., Reading (MA), 1978.
- [Be] A. S. Besicovitch, *Almost periodic functions*, Cambridge University Press, Cambridge, 1932.
- [B] H. Bohr, *Almost periodic functions*, Chelsea Pub. Co., New York, 1947.

- [C] A. Clark, *Linear flows on κ -solenoids*, to appear in Topology and its Applications.
- [MM] L. Markus and K. R. Meyer, *Periodic orbits and solenoids in generic Hamiltonian dynamical systems*, American Journal of Mathematics **102** (1980), No. 1, pp. 25–92.
- [NS] V.V. Nemytskii and V.V. Stepanov, *Qualitative Theory of Differential Equations*, Princeton Univ. Press, Princeton (NJ), 1960.
- [P] L. S. Pontryagin, *Topological groups*, Second Edition, Gordon and Breach, New York, 1966.
- [S] W. Scheffer, *Maps between topological groups that are homotopic to homomorphisms*, Proc. A.M.S. **33**, No. 2, 1972.

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